Midterm Exam - Optimization
B. Math III

25 February, 2026

(i) Duration of the exam is 2.5 hours.

(ii) The maximum number of points you can score in the exam is 75 (total = 80).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. (15 points) The support function, $S_K : S^{n-1} \to (-\infty, \infty]$, of a closed convex set $K \subseteq \mathbb{R}^n$ is defined as

$$S_K(\vec{y}) = \sup\{\langle \vec{y}, \vec{x} \rangle : \vec{x} \in K\} \leq \infty,$$

for every unit vector $\vec{y}$ in $\mathbb{R}^n$. Suppose that K and L are closed convex sets in $\mathbb{R}^n$. Show that $K = L$ if and only if $S_K = S_L$.

Total for Question 1: 15

2. A real (n, n)-matrix $A = ((\alpha_{ij}))$ is called *doubly stochastic* if $\alpha_{ij} \geq 0$ and $\sum_{k=1}^{n} \alpha_{kj} = \sum_{k=1}^{n} \alpha_{ik} = 1$ for $i, j \in \{1, \ldots, n\}$. A doubly stochastic matrix with components in $\{0, 1\}$ is called a permutation matrix.

   (a) (5 points) Prove that the set $K \subset \mathbb{R}^{n^2}$ of doubly stochastic matrices is compact and convex.

   (b) (15 points) Find, with justification, all extreme points of K.

Total for Question 2: 20

3. (15 points) Let $\mathbf{c}$ be a vector in $\mathbb{R}^n$. Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ where $\mathbf{x}$ varies over a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$. Prove that $\mathbf{y} \in \mathcal{P}$ is the unique optimal solution if and only if $\mathbf{c}^T\mathbf{d} > \mathbf{0}$ for every non-zero feasible direction $\mathbf{d}$ at $\mathbf{y}$.

Total for Question 3: 15

4. (10 points) Consider the problem

$$\begin{array}{rc} \text{minimize} & -x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

Total for Question 4: 10

5. Consider an LP in standard form with cost vector $\mathbf{c}$. Let $\mathbf{x}$ be a basic feasible solution associated with the basis matrix $\mathbf{B}$ and $\overline{\mathbf{c}}$ be the corresponding reduced cost vector.

   (a) (10 points) If $\overline{\mathbf{c}} \geq \mathbf{0}$, show that $\mathbf{x}$ is optimal.

   (b) (10 points) If $\mathbf{x}$ is optimal and nondegenerate, then show that $\overline{\mathbf{c}} \geq \mathbf{0}$.

Total for Question 5: 20